Greedy Method (FRIDAY)

APS 2 (SPIS 2016)

Algorithmic Problem Solving
Friday, August 12

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The Greedy Method (Example2)

You have the following chores to complete:

- Walk the dog (15 minutes)
- Mow the lawn (60 minutes)
- Shovel the snow (45 minutes)
- Take out the trash (2 minutes)
- Clean the pool (45 minutes)
- Wash the windows (75 minutes)
- Wash the car (30 minutes)
- Cook dinner (20 minutes)

You would like to complete as many jobs as you can in say 3 hours. Clearly describe an algorithm to do this. Discuss its correctness

The Greedy Method (Example3)

You have the following chores to complete and each pays:

- Walk the dog (15 minutes, $5)
- Mow the lawn (60 minutes, $50)
- Shovel the snow (45 minutes, $20)
- Take out the trash (2 minutes, $2)
- Clean the pool (45 minutes, $15)
- Wash the windows (75 minutes, $60)
- Wash the car (30 minutes, $15)
- Cook dinner (20 minutes, $5)

You would like to earn the most money possible in 2 hours. Clearly describe an algorithm to do this. Discuss its correctness.

The Greedy Method (Example4)

Suppose each one of you has a rumor. You can only share the rumor by passing a note.

The goal is to share the rumors so that everybody in the class knows all the rumors in the least possible number of note passes.

(you may write several rumors that you know in one note.)

Describe your algorithm and why it works.

Rumors

- What is your algorithm and how many notes were passed?

Rumors (lower bound.)

- Let m be the number of notes passed before everyone knows all the rumors.
- Claim: $m \geq 2n - 2$

- Proof of Claim: by induction.
- $n = 1$
- If there is one person then that person already knows all the rumors. so $m = 0 = 2(1) - 2$!!!

Rumors (lower bound.)

- Claim: $m \geq 2n - 2$

- Proof of Claim: by induction.
- Suppose for $n - 1$ people, $m \geq 2(n - 1) - 2 = 2n - 4$.

- Now, consider a new person comes in. That new person must contribute at the very least 2 more notes….
- One for sharing his rumor and one for receiving the rest of the rumors so…..
- for n people, $m \geq 2n - 4 + 2 = 2n - 2$!!!!!!!

Rumors (lower bound.)

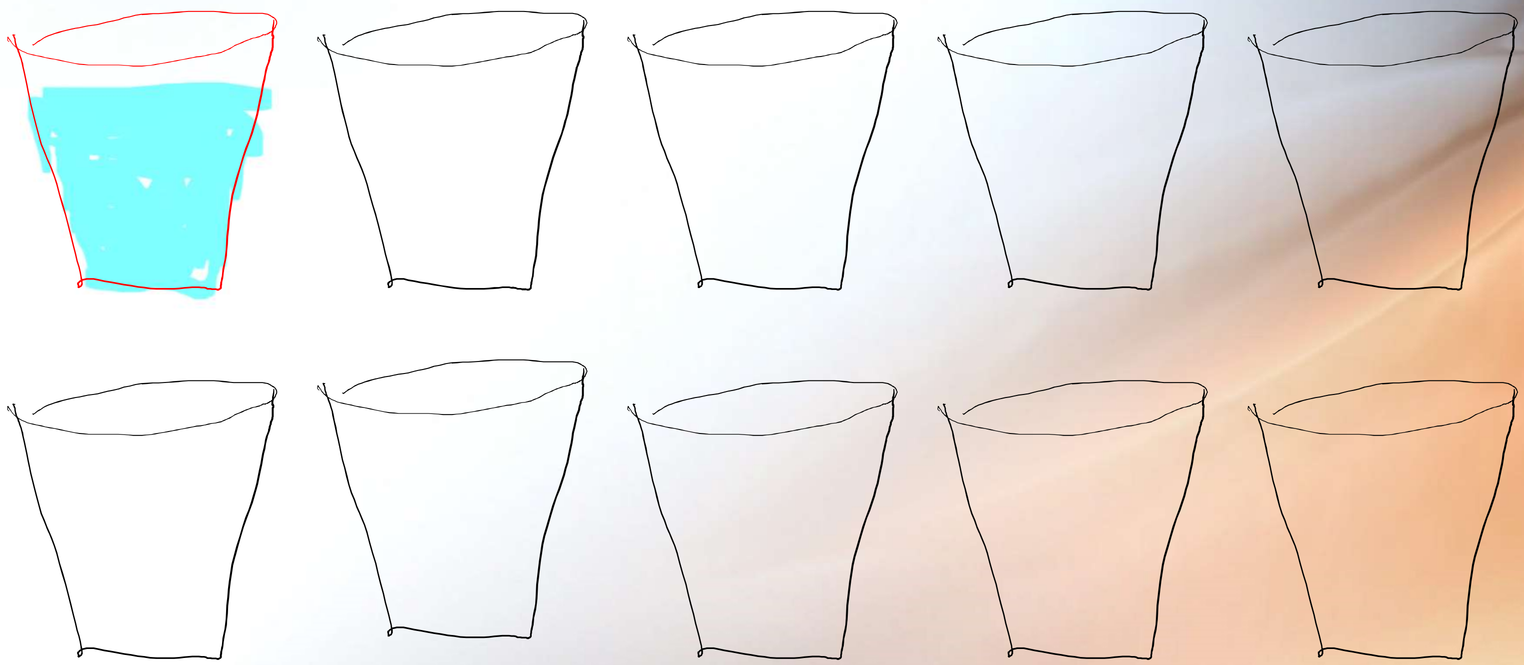
- Claim: $m \geq 2n - 2$

- Claim has been proven, and our algorithm achieves this lower bound, so our algorithm works!!!!!

The Greedy Method (Example5)

There are 10 identical cups, one of them with 100 pints of water and the others empty. You are allowed to perform the following operation: take two of the vessels and split the total amount of water in them equally between them. The object is to achieve a minimum amount of water in the original vessel (the one containing all the water in the initial setup) by a sequence of such operations. What is the best way to do this? You may do as many operations as you want. The goal is to minimize the amount of water in the original vessel.

The Greedy Method (Example5)



The Greedy Method (Example5)

A greedy approach:

Split the original vessel with an empty vessel. When there are no empty vessels you are done.

Why is this greedy?

How much is left in the original vessel?

The Greedy Method (Example5)

Why is this greedy?

Because you are making the maximum progress towards your goal at each step.

How much is left in the original vessel?

Since the original vessel has been halved 9 times, it now contains $100 \times \left(\frac{1}{2}\right)^9 \approx 0.195$ pints.

The Greedy Method (Proof)

As it turns out, the proof of this algorithm is more complicated than the actual algorithm.

The proof is presented in the homework notes.

I will walk you through it.

The Greedy Method (Proof)

The idea of the proof:

1. Set m to be the minimum amount of water in any vessel that has water.

2. Show that $m \geq 100/2^9$ (Show that there is a lower bound.)

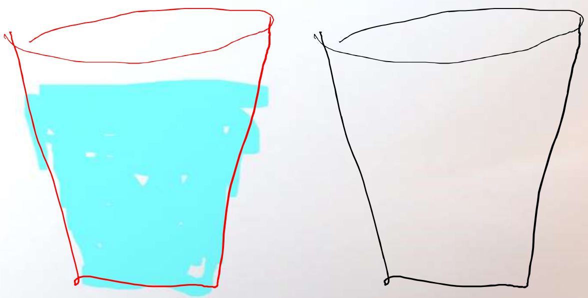
3. Conclude that our algorithm achieves that lower bound, therefore it returns the optimal solution.

The Greedy Method (Proof)

The proof:

1. Set m to be the minimum amount of water in any vessel that has water.

m starts out as 100. After the first step, m becomes 50.



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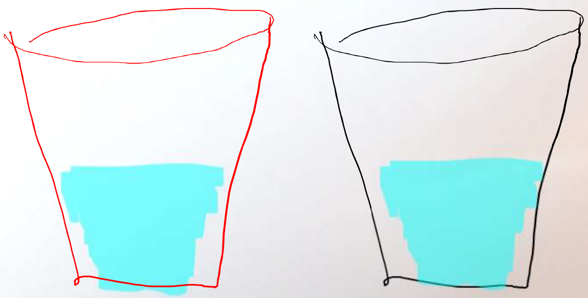
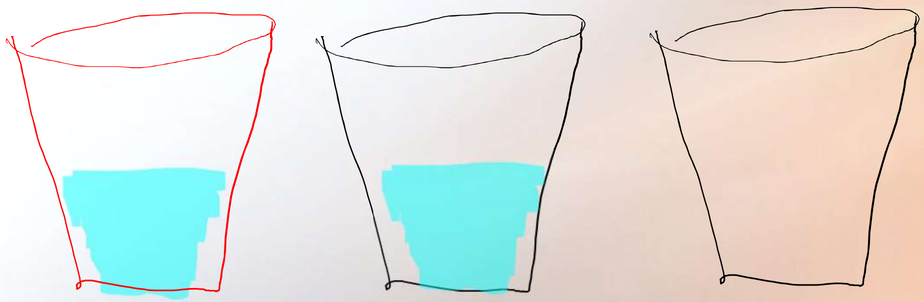


The Greedy Method (Proof)

m starts out as 100. After the first step, m becomes 50.

After the second step, m either stays at 50 or becomes 25.



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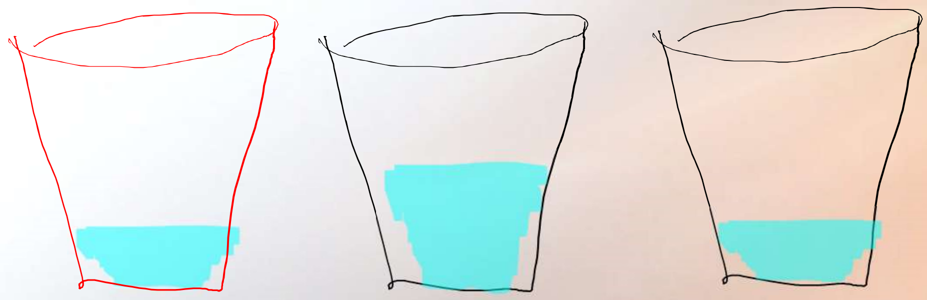

The Greedy Method (Proof)

Claim: The only way m can ever decrease is if a cup is split with an empty cup.

Proof of Claim: Suppose you split nonempty cups A and B each with w_A, w_B amount of water. Since m is the minimum amount of water in all non-empty cups, we have that

$$w_A \geq m, \qquad w_B \geq m$$

So when you split cup A and B, the result is

$$\frac{w_A + w_B}{2} \geq \frac{m+m}{2} = m$$

The Greedy Method (Proof)

Continuing with the proof, we now know that splitting a cup with an empty cup is the only way to possibly decrease m.

Suppose you go through a sequence of steps, (They may not be the greedy method.) And let i be the number of steps that split a non-empty cup with an empty cup.

(This sequence may include other steps.)

The Greedy Method (Proof)

Claim: After that sequence of steps, it must be the case that $m \geq 100/2^i$.

Proof of Claim:
Proof by induction on i.
$i = 1$: the first move has to be splitting the original with an empty cup leaving 50 pints in each so $m \geq \frac{100}{2^1} = 50.$

The Greedy Method (Proof)

Claim: After that sequence of steps, it must be the case that $m \geq 100/2^i$.

Proof of Claim:
Proof by induction on i.
Assume it is true for $i-1$, that no matter what sequence of splits that include splitting with $i-1$ empty vessels,

$$m \geq 100/2^{i-1}$$

Then after the ith split with an empty vessel, the most we can reduce m is by splitting the cup with m pints in it so

$$m \geq 100/2^{i-1}/2 = 100/2^i$$

The Greedy Method (Proof)

Claim: After that sequence of steps, it must be the case that $m \geq 100/2^i$.

The claim shows us that after we've split with 9 empty vessels, we must have $m \geq 100/2^9$. And it's not possible to ever have a sequence of actions that involves splitting with more than 9 empty vessels, since once an empty vessel is "used", it can never become empty again. Therefore the smallest possible value of m, the least amount of water that can ever be in any non-empty vessel, is $100/2^9$, and this is indeed what the greedy algorithm achieves.

The Greedy Method (Proof)

WHEW!!!

The Greedy Method (Proof)

WHEW!!!